Memo

To
To whom it may concern

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Subject
PID controller mass-spring-damper system

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Version control information

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Contents

Used references

- Berdahl and III (2007),
- Callafon (2014),
- Rowell (2004),
- Yon-Ping (*Mass-Damper-Spring systems/PID control of the simplest second-order systems*, eq. 23): PD-Controller, desired set-point will not be reached.
- Rao (2009, pg. 13, eq. 2-39): Integral of *Dirac delta functions* (unit impulse and unit step response)
- Åström and Murray (2016, pg. 47 eq. 2.19): Unit step response solution.
- Seborg et al. (2011, §8.6.1)

To Do

1 Transfer functions.

1 PID controller in SOBEK

The discretised PID-controller in SOBEK2 read:

$$f^n = f^{n-1} + K_p e^n + K_i \sum_{j=0}^{n} e^j + K_d \left(e^n - e^{n-1}\right), \tag{1}$$

No literature references are found for this PID-controller (i.e. Equation (1)).

The proposed discretised PID-controller read:

$$f^n = f^{n-1} + K_p \left(e^n - e^{n-1}\right) + K_i \, \Delta t_n \, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t} \tag{2}$$

This PID-controller (Equation (2)) is based on the linearisation of the standard PID-controller.

2 Aim of this document

The aim of this document is to find out what each term of the PID-controller does do to a general system. To reach that goal we look to the Mass-Spring-Damper system without loss of generality.

3 Conclusion

Mathematical physics

- Proportional gain factor: influence on equilibrium state. But it does not reach the required set-point.
- Integral gain factor: influence on the equilibrium state. You need this term to reach the equilibrium state.
- Derivative gain factor: Influence transition time to the equilibrium state, but it does not influence the equilibrium state.

Numerical experiments

Some numerical experiments are performed. Varying time step and varying the time integration method of the PID-controller. The time integration of the Mass-Spring-Damper-system is implicit.

- Explicit implementation (black box approach of the PID-controller) does have a severe draw back on the computation time because the Δt should be decreased w.r.t. the other methods.

4 Mass-Spring-Damper system

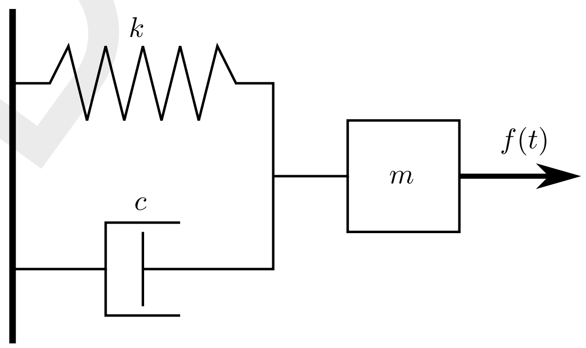

Figure 1: Drawing of Mass-Spring-Damper system.

Equation of Mass-Spring-Damper system ($m > 0$, $c > 0$ and $k > 0$):

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad \Leftrightarrow \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f(t) \tag{3}$$

The natural (free) angular velocity ω_n is ($c = 0$ and $f(t) = 0$):

$$\omega_n = \sqrt{\frac{k}{m}}, \tag{4}$$

The decay time constant τ is:

$$\tau = \frac{2m}{c} \tag{5}$$

and the damping ratio ζ is:

$$\zeta = \frac{c}{2\sqrt{mk}} \tag{6}$$

The damping ratio is related to ω_n.

$mr^2 + cr + k = 0 \Rightarrow r_{1,2} = \frac{-c\pm\sqrt{c^2-4mk}}{2m}$
Decay of solution is: $\exp\left(\frac{-c}{2m}\right)$, so the decay time constant is: $\tau = \frac{2m}{c}$
Damping rate is: $\exp\left(-\zeta\omega_n t\right) = \exp\left(\frac{-ct}{2m}\right) \Rightarrow \zeta\omega_n = \frac{c}{2m} \Rightarrow \zeta = \frac{c}{2\omega_n m} \Rightarrow \zeta = \frac{c}{2\sqrt{km}}$.
The equation can also be written as:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 = 0 \tag{7}$$

$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0 \Rightarrow r_{1,2} = \zeta\omega_n \pm \frac{1}{2}\sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2} \Rightarrow r_{1,2} = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$.
Define $\omega_d = \omega_n\sqrt{1-\zeta^2}$ as the frequency for the under damped oscillations.
$\zeta = 0$ not damped (oscillations)
$\zeta < 1$ under damped (oscillations)
$\zeta = 1$ critical damped
$\zeta > 1$ over damped

4.1 Energy of Mass-Spring-Damper system

Write the Mass-Spring-Damper system as a set of first order of PDE's, without external forces as:

$$\dot{x} = v \tag{8}$$

$$m\dot{v} = -cv - kx, \tag{9}$$

The total energy of the system read:

$$E_{tot} = E_{kin} + E_{pot} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{10}$$

Energy conserving:

$$\frac{d}{dt}E_{tot} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) \tag{11}$$

$$= mv\dot{v} + kx\dot{x} \tag{12}$$

Substituting gives:

$$\frac{d}{dt}E_{tot} = mv\frac{-cv-kx}{m} + kxv = -cv^2 - vkx + kxv = -cv^2 \tag{13}$$

So if $c = 0$ the system is energy conserving, other wise the system is dissipative. If $\frac{d}{dt}E_{tot} = 0$ the Mass-Spring-Damper system is called a Hamiltonian system, if $\frac{d}{dt}E_{tot} \leq 0$ the Mass-Spring-Damper system is called a Lyapunov function.

5 Feedback loop, PID-controller

Implement the feed back loop (external force is dependent on solution, Berdahl and III (2007)) and an external force as step function (i.e. new setpoint)

$$f(t) = -K_p x - K_d \dot{x} + u(t) \tag{14}$$

with

K_p	gain factor proportional to x.
K_d	gain factor proportional to the time derivative of x (i.e. $\dot{x}$, velocity) and
$u(t)$	the unit step function is defined as Rao (2009, pg. 13, eq. 2-28):

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{15}$$

After substitution of Equation (14) in Equation (3) (consider only $t \geq 0$; assume everything is in rest for $t < 0$) we get:

$$m\ddot{x} + c\dot{x} + kx = -K_p x - K_d \dot{x} + u(t) \tag{16}$$

$$m\ddot{x} + (c + K_d)\dot{x} + (k + K_p)x = u(t) \tag{17}$$

To evaluate the stepfunction $u(t)$ of Equation (15), we first discuss the solution for the *Dirac delta function* (or *unit impulse function*) and than the *unit step function*.

5.1 Unit impulse function

To evaluate the *Dirac delta function* (or *unit impulse function*):

$$\delta(t - a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases} \tag{18}$$

which satisfies

$$\int_{-\infty}^{\infty} \delta(t - a) dt = 1 \tag{19}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - a) dt = f(a) \tag{20}$$

Suppose $g(t)$ is the solution of the system:

$$m\ddot{g} + c\dot{g} + kg = \delta(t) \tag{21}$$

Integrate this equation from 0 to T ($T > 0$), just the length of the pulse. Resulting in a valid value at the right hand side

$$\int_0^T (m\ddot{g} + c\dot{g} + kg)\, dt = \int_0^T \delta(t)\, dt \tag{22}$$

$$\int_0^T m\ddot{g}\, dt + \int_0^T c\dot{g}\, dt + \int_0^T kg\, dt = \int_0^T \delta(t)\, dt \tag{23}$$

Taking the limit as $T \downarrow 0$, we obtain (term by term):

$$\lim_{T\downarrow 0} \int_0^T m\ddot{g}(t)\,dt = \lim_{T\downarrow 0} m\dot{g}(t)|_0^T = \lim_{T\downarrow 0} m\left[\dot{g}(T) - \dot{g}(0)\right] = m\dot{g}(0^+) \qquad \dot{g}(t) \text{ is discontinue} \tag{24}$$

$$\lim_{T\downarrow 0} \int_0^T c\dot{g}(t)\,dt = \lim_{T\downarrow 0} cg(t)|_0^T = \lim_{T\downarrow 0} c\left[g(T) - g(0)\right] = 0 \qquad g(t) \text{ is continue} \tag{25}$$

$$\lim_{T\downarrow 0} \int_0^T kg(t)\,dt = \lim_{T\downarrow 0} ktg(0)|_0^T = \lim_{T\downarrow 0} k\left[Tg(0) - 0g(0)\right] = 0 \qquad \text{because } g(0) = 0 \tag{26}$$

$$\int_0^T \delta(t)\,dt = 1 \tag{27}$$

and thus

$$m\dot{g}(0^+) = 1 \quad \Rightarrow \quad \dot{g}(0^+) = \frac{1}{m} \tag{28}$$

Now that we know that the response of a second-order resting system is to change the velocity (while leaving position unchanged), we can use this fact to obtain the impulse response $g(t)$. In particular, assuming an underdamped system, we know that the general form of the free response is given as

$$g(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \tag{29}$$

Therefore, with $g(0) = 0$ and $\dot{g}(0^+) = 1/m$ the response of the system to a unit impulse at $t = 0$ is given as

$$g(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{m\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t & t > 0 \end{cases} \tag{30}$$

5.2 Unit step function

The unit step function is defined as

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} \tag{31}$$

The relation between the $\delta(t)$ and $u(t)$ is as follows:

$$u(t-a) = \int_{-\infty}^{t} \delta(\tau - a)\,d\tau \tag{32}$$

$$\frac{d\,u(t-a)}{dt} = \delta(\tau - a) \tag{33}$$

The function $s(t)$ is called the *step response* and satisfies:

$$m\ddot{s} + c\dot{s} + ks = u(t) \tag{34}$$

The function $s(t)$ will be obtained as follows, starting from the unit impulse solution $g(t)$:

$$m\ddot{g} + c\dot{g} + kg = \delta(t) \tag{35}$$

$$m\ddot{g} + c\dot{g} + kg = \frac{d\,u(t)}{dt} \tag{36}$$

Integrating this equation from $-\infty$ to t:

$$\int_{-\infty}^{t} \left(m\frac{d^2\,g}{d\tau^2} + c\frac{d\,g}{d\tau} + kg \right) d\tau = \int_{-\infty}^{t} \frac{d\,u(\tau)}{d\tau}\, d\tau = u(t) \tag{37}$$

Using the fundamental theorem of calculus we get

$$\left(m\frac{d^2}{dt^2} + c\frac{d}{dt} + k \right) \int_{-\infty}^{t} g(\tau)\, d\tau = u(t) \tag{38}$$

It is seen that

$$s(t) = \int_{-\infty}^{t} g(\tau)\, d\tau \tag{39}$$

For $t \leq 0$ we have $s(t) = 0$ and for $t > 0$ we have (integrating Equation (30))

$$s(t) = \int_{0}^{t} \left(\frac{1}{m\omega_d} e^{-\zeta\omega_n\tau} \sin\omega_d\tau \right) d\tau \tag{40}$$

After some calculation, using $\sin x = (e^{ix} - e^{-ix})/2i$ and $(e^{ix} + e^{-ix})/2 = \cos x$ (Rao, 2009), we get

$$s(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{m\omega_n^2} \left(1 - e^{-\zeta\omega_n t} \left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin\omega_d t \right) \right) & t > 0 \end{cases} \tag{41}$$

or

$$s(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{m\omega_n^2} \left(1 - e^{-\zeta\omega_n t} \left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t \right) \right) & t > 0 \end{cases} \tag{42}$$

This solution brings the system to a new equilibrium state (using Equation (4)):

$$\lim_{t\to\infty} s(t) = \frac{1}{m\omega_n^2} = \frac{1}{k} \tag{43}$$

6 PID controller (positional)

Assume that the equilibrium value x_∞ or setpoint (new equilibrium) is a desired value, the error (deviation) to that value is defined as: $e(t) = x_\infty - x(t)$. This error (deviation) will be investigated.

The PID-controller can been seen as an external force on the Mass-Spring-Damper-system and read:

$$f(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \frac{de(t)}{dt}, \tag{44}$$

where

K_p	gain factor proportional to $e(t)$,
K_i	gain factor proportional to the time integral of $e(t)$ and
K_d	gain factor proportional to the time derivative of $e(t)$.

6.1 Proportional term

Consider just the proportional term, so $K_i = K_d = 0$.

The proportial term $K_p e(t)$ does influence the equilibrium state of the Mass-Spring-Damper system, but will not reached the desired equilibrium state x_∞.

$$m\ddot{x} + c\dot{x} + kx = K_p(x_\infty - x) \tag{45}$$
$$m\ddot{x} + c\dot{x} + (k + K_p)x = K_p x_\infty \tag{46}$$

So the particular solution will **not** reach the equilibrium state ($\lim_{t\to\infty}$, and $\ddot{x} = \dot{x} = 0$):

$$\lim_{t\to\infty} x(t) = \frac{K_p}{(k + K_p)} x_\infty \quad \text{and} \quad k + K_p > 0 \tag{47}$$

also the free angular frequency is influenced by the proportional gain K_p

$$\omega_n = \sqrt{\frac{k + K_p}{m}} \quad \text{and} \quad k + K_p > 0 \tag{48}$$

6.2 Derivative term

Consider just the derivative term, so $K_p = K_i = 0$.

The derivative term $K_d \dot{e}(t)$ does **not** influence the equilibrium state of the Mass-Spring-Damper system:

$$m\ddot{x} + c\dot{x} + kx = K_d \frac{d}{dt}(x_\infty - x) \tag{49}$$
$$m\ddot{x} + c\dot{x} + kx = K_d(\dot{x}_\infty - \dot{x}) \tag{50}$$

With $\dot{x}_\infty = 0$ for the equilibrium state, so

$$m\ddot{x} + c\dot{x} + kx = -K_d \dot{x} \tag{51}$$
$$m\ddot{x} + (c + K_d)\dot{x} + kx = 0 \tag{52}$$

So, just the damping factor (resistance, friction) is adjusted by the derivative gain K_d and the equilibrium state is the same as for the homogeneous solution ($\ddot{x} = \dot{x} = 0$):

$$\lim_{t\to\infty} kx(t) = 0 \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0. \tag{53}$$

Remark: $c + K_d > 0$ other wise the system is unstable, see section 4.1

6.3 Integral term

Consider just the integral term, so $K_p = K_d = 0$.

The integral term $K_i \int_0^t e(\tau)\, d\tau$ does influence the equilibrium state of the Mass-Spring-Damper system. Switching a Mass-Spring-Damper system to another equilibrium state is done by prescribing a value other than zero as external force. For example a constant value, the integral of a *Dirac delta function* or another bounded integral. Prescribing a constant value of 1 will give as equilibrium state ($\ddot{x} = \dot{x} = 0$):

$$\lim_{t\to\infty} (m\ddot{x} + c\dot{x} + kx) = 1 \tag{54}$$

$$kx = 1 \quad \Rightarrow \quad x = \frac{1}{k} \tag{55}$$

The integral of the *Dirac delta function* is equal to 1,

$$\int_{-\infty}^{\infty} \delta(t-a)dt = 1 \tag{56}$$

and will give therefore the same solution as above (Equation (55)).

Prescribing the external force as

$$f(t) = \int_0^t (x_\infty - x(\tau))\, d\tau \tag{57}$$

will force the solution $x(t)$ to x_∞, because

$$\lim_{t\to\infty} (x_\infty - x(t)) = 0 \tag{58}$$

and the integral is bounded ($< \infty$).

7 PID controller (velocity)

Assume that a equilibrium value x_∞ or setpoint (a new equlibrium), is a desired value, the error (deviation) to that value is defined as: $e(t) = x_\infty - x(t)$, and need to be investigated.

The PID-controller can been seen as a external force on the Mass-Spring-Damper-system and read:

$$f_{\text{PID}}(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \frac{de(t)}{dt}, \quad \text{equal to Equation (44)} \tag{59}$$

The time derivative (velocity) of the PID-controller (Equation (59)) read:

$$\frac{\partial f_{\text{PID}}}{\partial t} = K_p \frac{de(t)}{dt} + K_i e(t) + K_d \frac{d^2 e(t)}{dt^2} \tag{60}$$

Which comes from the linearisation (i.e. $\delta y = \mathcal{K}\, \delta x$) of the PID-controller:

$$f_{\text{PID}}(t + \delta t) = f_{\text{PID}}(t) + \mathcal{K}\, \delta t \tag{61}$$

$$f_{\text{PID}}(t + \delta t) - f_{\text{PID}}(t) = \mathcal{K}\, \delta t \tag{62}$$

Divide Equation (62) by δt and take $\lim_{\delta t \to 0}$, we get:

$$\lim_{\delta t \to 0} \frac{f_{\text{PID}}(t + \delta t) - f_{\text{PID}}(t)}{\delta t} = \mathcal{K} \tag{63}$$

$$\frac{\partial f_{\text{PID}}}{\partial t} = \mathcal{K} \equiv \text{RHS Equation (60)} \tag{64}$$

In discrete form Equation (60) read, with $\Delta t_n = \Delta t$ is constant, $e^n = x_\infty - x^n$:

$$\frac{f^n - f^{n-1}}{\Delta t_n} = K_p \frac{e^n - e^{n-1}}{\Delta t_n} + K_i\, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t^2} \tag{65}$$

$$f^n = f^{n-1} + \Delta t_n \left\{ K_p \frac{e^n - e^{n-1}}{\Delta t_n} + K_i\, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t^2} \right\} \tag{66}$$

$$f^n = f^{n-1} + K_p \left(e^n - e^{n-1} \right) + K_i\, \Delta t_n\, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t} \tag{67}$$

8 Determine coefficients from experiments

Taken from Callafon (2014)

Estimation of model parameters

With the times t_0, t_n and the values y_0, y_n and y_∞ from step response:

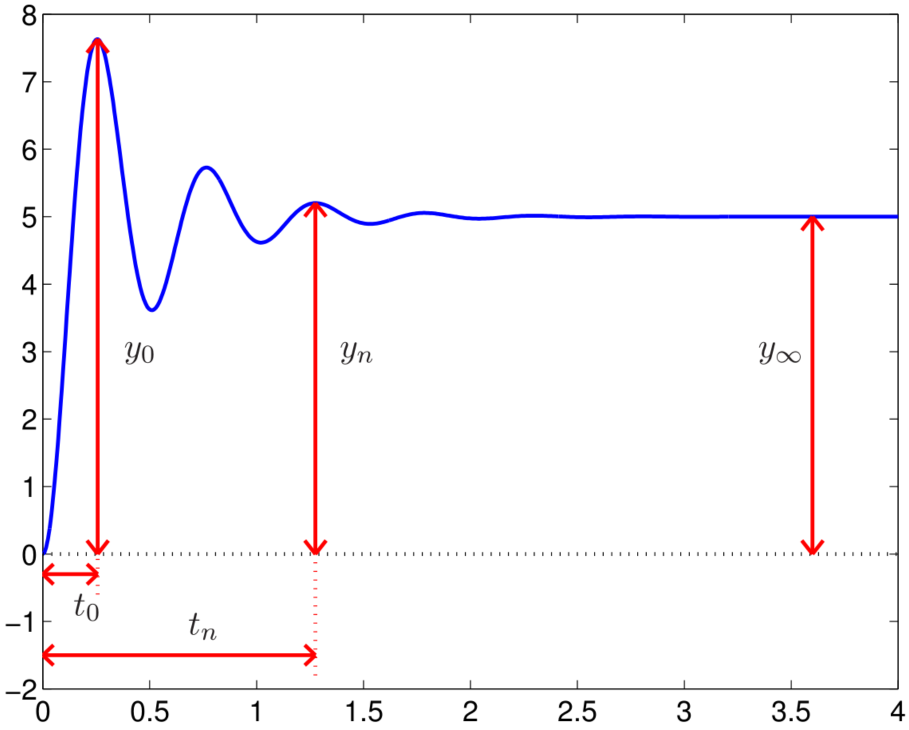

Figure 2: Collect data from measurement.

Allows us to estimate:

$$\widehat{\omega}_d = 2\pi \frac{n}{t_n - t_0} \quad \text{damped resonance frequency} \tag{68}$$

$$\widehat{\beta\omega_n} = \frac{1}{t_n - t_0} \ln\left(\frac{y_0 - y_\infty}{y_0 - y_\infty}\right) \quad \text{exponential decay term} \tag{69}$$

where n as subscript of t is the *number of oscillations* between t_n and t_0. With the estimations of the demped frequency $\widehat{\omega}_d$ and $\widehat{\beta\omega_n}$ we can now compute the natural frequence ω_n and the damping ratio β:

$$\widehat{\omega}_n = \sqrt{\widehat{\omega}_d^2 + \widehat{\beta\omega_n}^2} \quad \text{undamped resonance frequency} \tag{70}$$

$$\beta = \frac{\widehat{\beta\omega_n}}{\widehat{\omega}_n} \quad \text{damping ratio} \tag{71}$$

9 Numerical discretisation

9.1 Mass-Spring-Damper system as system of first order PDE's

The equation of a Mass-Spring-Damper system

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{72}$$

can be written as a system of first order partial differential equations, which read:

$$\dot{x} = v \tag{73}$$

$$m\dot{v} = -cv - kx + f(t) \tag{74}$$

with initial conditions: $\dot{x} = v = 0$ and $x = 1$. In matrix notation this equation read:

$$\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \tag{75}$$

This system of equations in discrete form, with $\Delta t_{n+1} = t^{n+1} - t^n$, read:

$$\frac{x^{n+1} - x^n}{\Delta t_{n+1}} = v^n \tag{76}$$

$$m\frac{v^{n+1} - v^n}{\Delta t_{n+1}} = -cv^{n+1} - kx^{n+1} + f(x^{n+1}, x^n, \ldots) \tag{77}$$

Rearranging gives:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{78}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} kx^{n+1} + \frac{\Delta t_{n+1}}{m} f(x^{n+1}, x^n, \ldots) \tag{79}$$

9.2 PID controller (positional)

9.2.1 Implicit Mass-Spring-Damper system with explicit PID-controller

An explicit discrete form of the PID-controller (Equation (44)) read:

$$f^n = K_p e^n + K_i \sum_{j=0}^{n} e^j \Delta t_j + K_d \frac{e^n - e^{n-1}}{\Delta t_n} \tag{80}$$

The system of equations will than read:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{81}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} kx^{n+1} + \frac{\Delta t_{n+1}}{m} \left[K_p e^n + K_i \sum_{j=0}^{n} e^j \Delta t_j + K_d \frac{e^n - e^{n-1}}{\Delta t_n} \right] \tag{82}$$

9.2.2 Implicit Mass-Spring-Damper system with implicit PID-controller

An implicit discrete form of the PID-controller (Equation (44)) read:

$$f^{n+1} = K_p e^{n+1} + K_i \sum_{j=0}^{n+1} e^j \Delta t_j + K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} \tag{83}$$

The system of equations will than read:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{84}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} k x^{n+1} + \frac{\Delta t_{n+1}}{m} \left[K_p e^{n+1} + K_i \sum_{j=0}^{n+1} e^j \Delta t_j + K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} \right] \tag{85}$$

9.2.3 Corrected Mass-Spring-Damper system while using an explicit PID controller

Therefore we will adjust the Mass-Spring-Damper system for the explicit values used by the PID-controller by adding those terms who will make the PID-controller implicit.

We will separate Equation (80) from Equation (83). The individual terms belonging to Equation (80) will be placed between square brackets.

$$f^{n+1} = K_p e^{n+1} + K_i \sum_{j=0}^{n+1} e^j \Delta t_j + K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} \tag{86}$$

$$\begin{aligned} &= K_p e^{n+1} - K_p e^n + [K_p e^n] + \\ &K_i e^{n+1} \Delta t_{n+1} + \left[K_i \sum_{j=0}^{n} e^j \Delta t_j \right] \\ &+ K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} - K_d \frac{e^n - e^{n-1}}{\Delta t_n} + \left[K_d \frac{e^n - e^{n-1}}{\Delta t_n} \right] \end{aligned} \tag{87}$$

The part between square brackets, is equal to Equation (80), which is the PID-controller based on explicit time levels:

$$f^n = K_p e^n + K_i \sum_{j=0}^{n} e^j \Delta t_j + K_d \frac{e^n - e^{n-1}}{\Delta t_n} \tag{88}$$

So we get:

$$f^{n+1} = K_p e^{n+1} - K_p e^n + K_i e^{n+1} \Delta t_{n+1} + K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} - K_d \frac{e^n - e^{n-1}}{\Delta t_n} + f^n \tag{89}$$

Substitution of Equation (103) in Equation (79) leads to the following system of equations:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{90}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} k x^{n+1} + \Delta t_{n+1} \frac{1}{m} \left[-K_p (x^{n+1} - x^n) + K_i e^{n+1} \Delta t_{n+1} + K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} - K_d \frac{e^n - e^{n-1}}{\Delta t_n} + f^n \right] \tag{91}$$

Some rearranging:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{92}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} (k + K_p) x^{n+1} + \frac{\Delta t_{n+1}}{m} K_p x^n + + \frac{\Delta t_{n+1}}{m} K_i e^{n+1} \Delta t_{n+1} + + \frac{\Delta t_{n+1}}{m} K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} - \frac{\Delta t_{n+1}}{m} K_d \frac{e^n - e^{n-1}}{\Delta t_n} + \frac{\Delta t_{n+1}}{m} f^n \tag{93}$$

f^n is given by Equation (80).

9.3 PID controller (velocity)

9.3.1 Implicit Mass-Spring-Damper system with explicit velocity PID controller

An explicit discrete form of the PID-controller based on the linearisation (Equation (60)) read:

$$f^n = f^{n-1} + K_p \left(e^n - e^{n-1}\right) + K_i \, \Delta t \, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t}, \quad \text{equal to Equation (67)} \tag{94}$$

The system of equations will than read:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{95}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} k x^{n+1} + \frac{\Delta t_{n+1}}{m} \left[f^{n-1} + K_p \left(e^n - e^{n-1}\right) + K_i \, \Delta t \, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t} \right] \tag{96}$$

9.3.2 Implicit Mass-Spring-Damper system with implicit velocity PID controller

An implicit discrete form of the PID-controller based on the linearisation (Equation (60)) read:

$$f^{n+1} = f^n + K_p \left(e^{n+1} - e^n\right) + K_i \, \Delta t \, e^{n+1} + K_d \frac{e^{n+1} - 2e^n + e^{n-1}}{\Delta t}, \quad \text{equal to Equation (67)} \tag{97}$$

The system of equations will than read:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{98}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} k x^{n+1} + \frac{\Delta t_{n+1}}{m}\left[f^n + K_p\left(e^{n+1} - e^n\right) + K_i\,\Delta t\, e^{n+1} + K_d \frac{e^{n+1} - 2e^n + e^{n-1}}{\Delta t}\right] \tag{99}$$

9.3.3 Corrected Mass-Spring-Damper system while using an explicit velocity PID controller

Therefore we will adjust the Mass-Spring-Damper system for the explicit values used by the velocity PID-controller by adding those terms who will make the velocity PID-controller implicit.

We will separate Equation (94) from Equation (97). The individual terms belonging to Equation (94) will be placed between square brackets. Δt is constant.

$$f^{n+1} = f^n + K_p\left(e^{n+1} - e^n\right) + K_i\,\Delta t\, e^{n+1} + K_d \frac{e^{n+1} - 2e^n + e^{n-1}}{\Delta t} \tag{100}$$

$$\begin{aligned} &= f^n - f^{n-1} + \left[f^{n-1}\right] + \\ &+ K_p\left(e^{n+1} - 2e^n + e^{n-1}\right) + \left[K_p\left(e^n - e^{n-1}\right)\right] \\ &+ K_i\, e^{n+1}\Delta t_{n+1} - K_i e^n \Delta t_n + \left[K_i\, e^n \Delta t_n\right] \\ &+ \ldots + \left[K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t}\right] \end{aligned} \tag{101}$$

The part between square brackets, is equal to Equation (94), which is the PID-controller based on explicit time levels:

$$f^n = f^{n-1} + K_p\left(e^n - e^{n-1}\right) + K_i\,\Delta t\, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t} \tag{102}$$

So we get:

$$f^{n+1} = \ldots + f^n \tag{103}$$

Substitution of Equation (103) in Equation (79) leads to the following system of equations:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{104}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} k x^{n+1} + \frac{\Delta t_{n+1}}{m}\left(\ldots + f^n\right) \tag{105}$$

Some rearranging:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{106}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n + \ldots + \frac{\Delta t_{n+1}}{m} f^n \tag{107}$$

f^n is given by Equation (80).

10 Experiments

Mass-Spring-Damper-system:

$$m\ddot{x} + c\dot{x} + kx = f(t), \tag{108}$$

$$f(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \frac{de(t)}{dt} \tag{109}$$

with

Initial values: $x(0) = 1$ and $\dot{x}(0) = 0$.
Coefficients: $m = 100$, $c = 2$, $k = 1$.
PID gain factors: $K_p = 10.0$; $K_i = 0.05$; $K_d = 10.0$.
Setpoint: 0.5.

$e(t) = x_\infty - x(t)$.
$x_\infty =$ setpoint.

Table 1: Equilibrium solution.

K_p	K_i	K_d	Equili
10.0	0.0	0.0	0.45
0.0	0.05	0.0	0.5
0.0	0.0	10.0	0.0

10.1 Solutions determined by Maplesoft

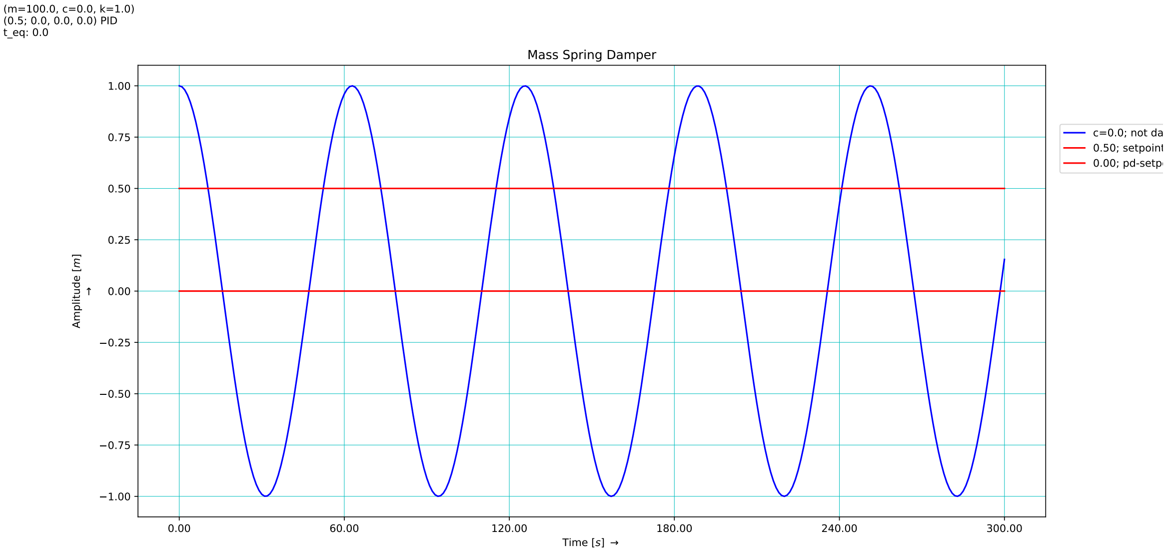

Figure 3: The natural frequency; $c = K_p = K_i = K_d = 0.$

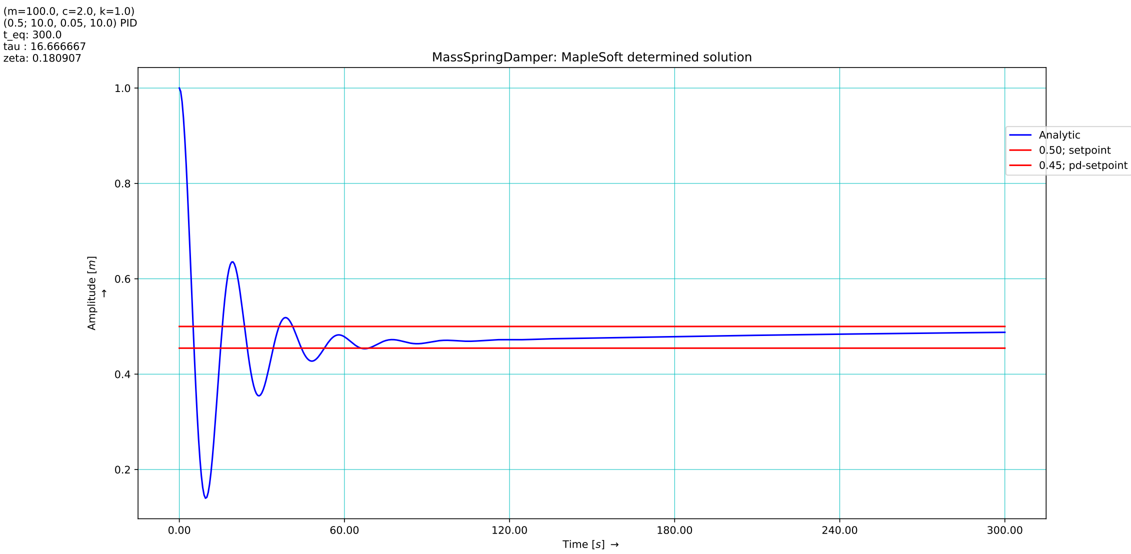

Figure 4: PID controller on Mass-Spring-Damper using MapleSoft solution. The integral term is quite small and therefor the influence is noticed after a while.

10.2 Numerical experiments

Table 2: Performed numerical experiments.

time integration	Δt $1.0\ s$	Δt $0.5\ s$	Δt $0.25\ s$
explicit (§9.2.1)	✓	✓	✓
implicit (§9.3.2)	✓	✓	✓
corrected (§9.2.3)	✓	✓	✓

10.2.1 Time integration method

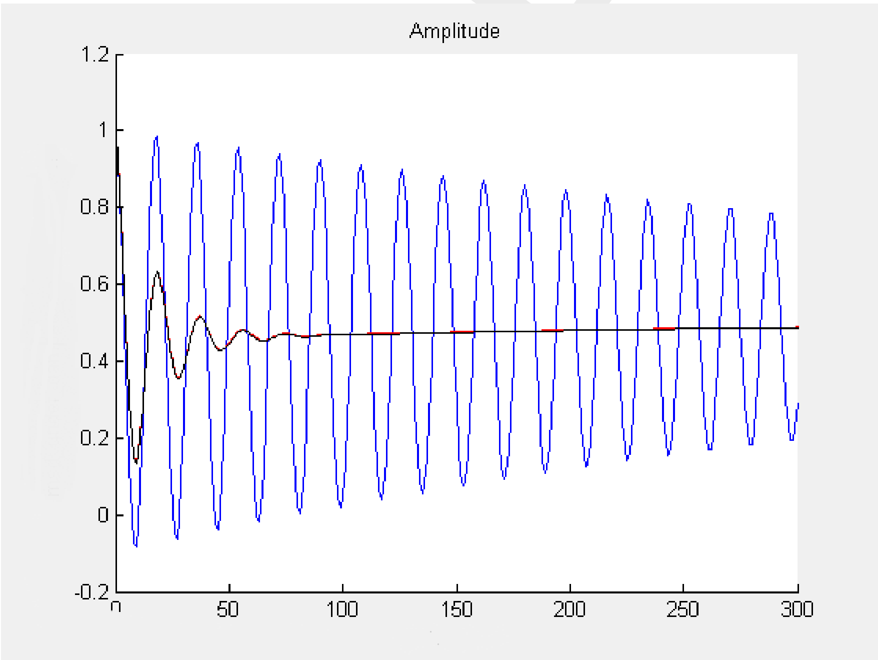

Figure 5: Numerical experiment ($\Delta t = 1\ s$); explicit (blue, §9.2.1), implicit (red, §9.3.2), corrected (black, §9.2.3).

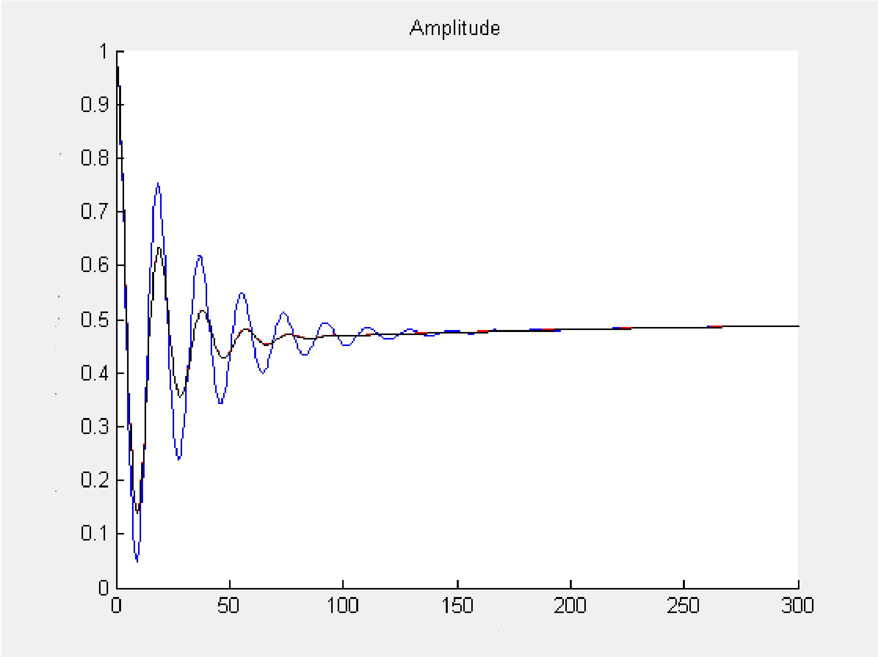

Figure 6: Numerical experiment ($\Delta t = 0.5\ s$); explicit (blue, §9.2.1), implicit (red, §9.3.2), corrected (black, §9.2.3).

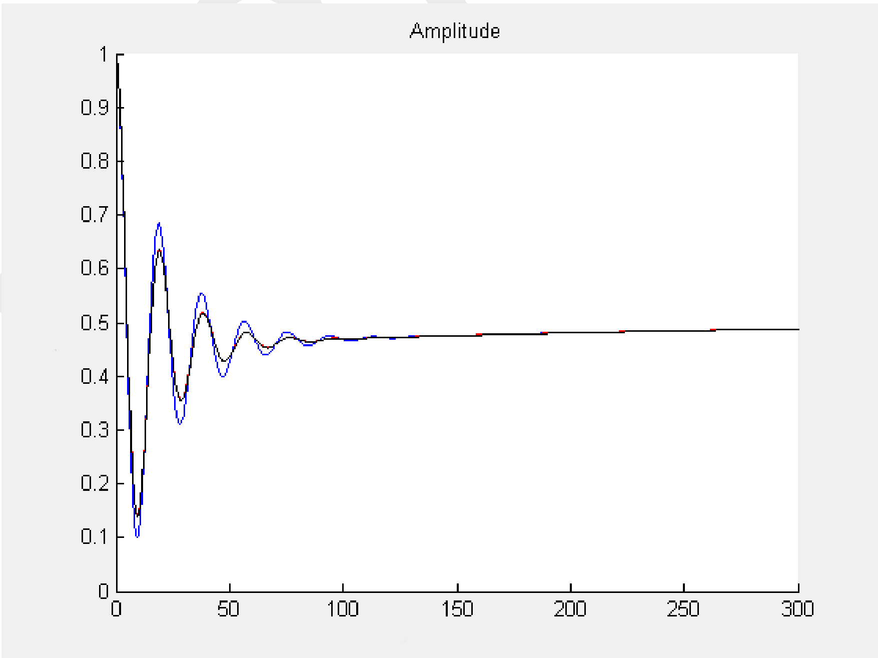

Figure 7: Numerical experiment ($\Delta t = 0.25\ s$); explicit (blue, §9.2.1), implicit (red, §9.3.2), corrected (black, §9.2.3).

10.2.2 Convergence behaviour

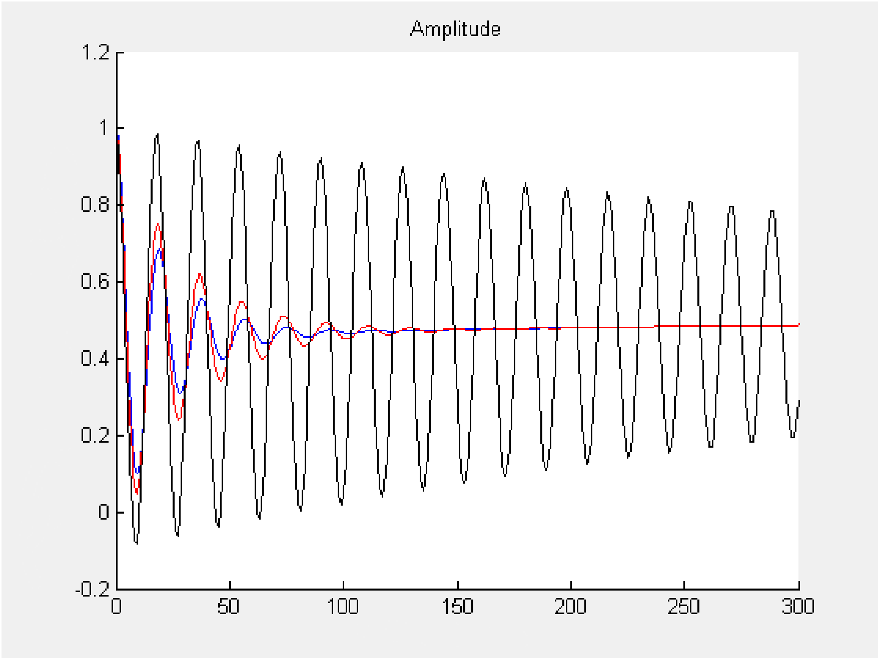

Figure 8: Numerical experiment (explicit §9.2.1); black, $\Delta t = 1.0\ s$; red, $\Delta t = 0.5\ s$; blue, $\Delta t = 0.25\ s$.

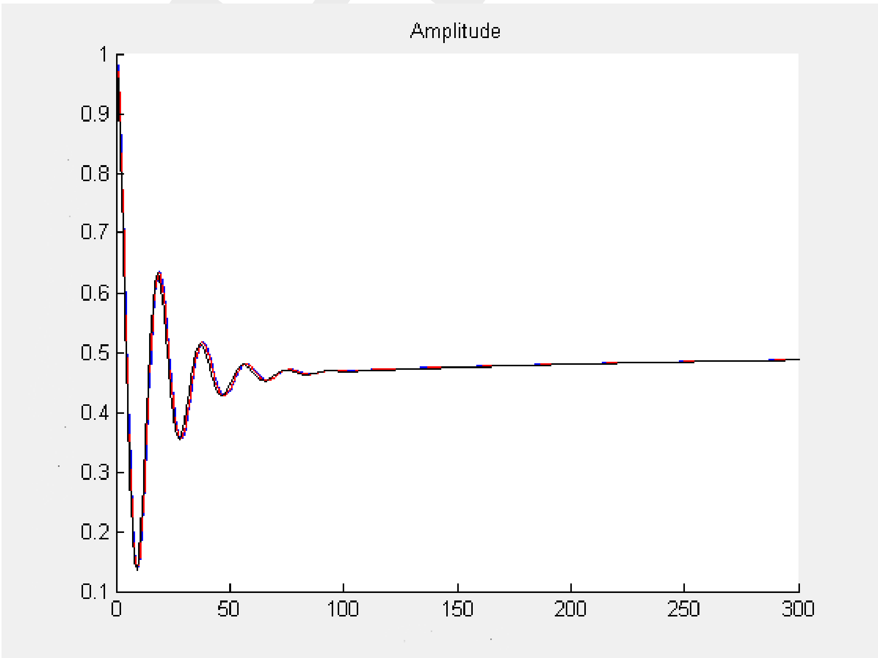

Figure 9: Numerical experiment (corrected §9.2.3); black, $\Delta t = 1.0\ s$; red, $\Delta t = 0.5\ s$; blue, $\Delta t = 0.25\ s$.

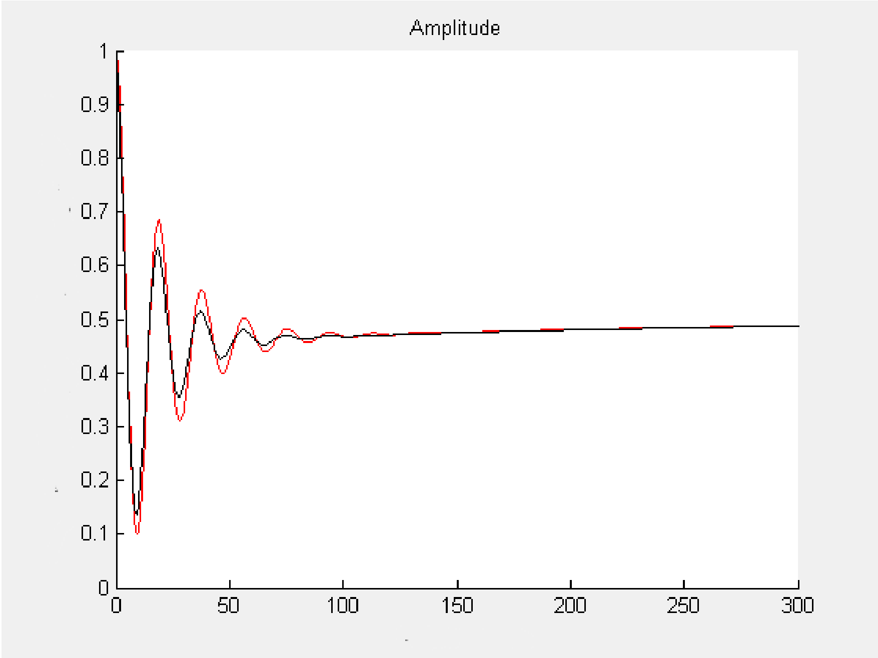

Figure 10: Numerical experiment, different Δt and time integration; explicit (red, $\Delta t = 0.25\ s$, §9.2.1), corrected (black, $\Delta t = 1.0\ s$, §9.2.3).

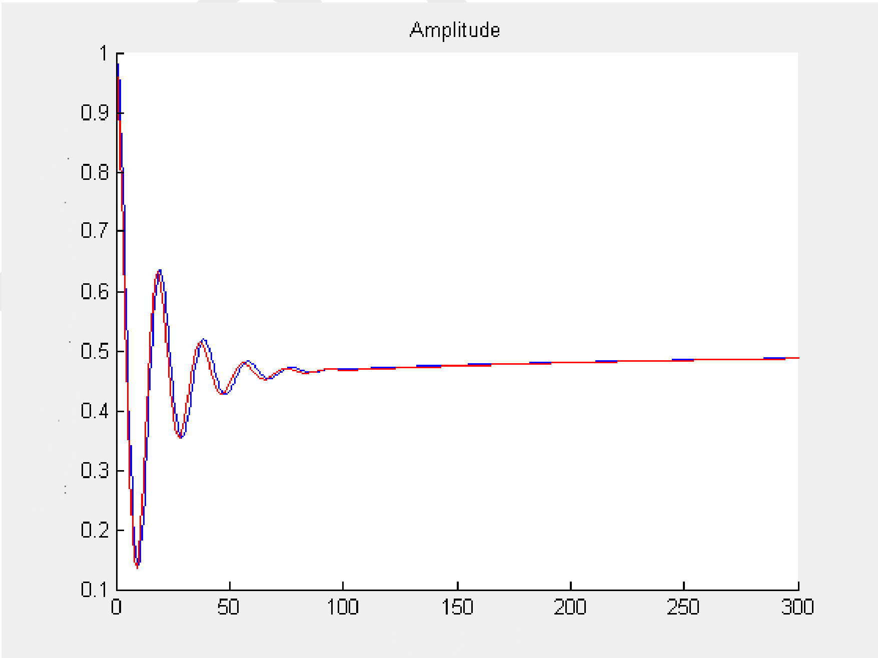

Figure 11: Maple solution (blue) vs corrected (red, $\Delta t = 1.0\ s$, §9.2.3).

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